PARTICLE-LIKE SOLUTIONS OF A NONLINEAR WAVE EQUATION

(CHASTITSEPODOBNYE RESHENIIA NELINEINOGO VOLNOVOGO URAVNENIIA)

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The Liapunov stability of solutions of the form (1.2) of the nonlinear wave equation (1.1) is considered. Hypotheses on the behavior of one class of solutions of equation (1.1) for $t \to \infty$ are given which results from the representation of the irreversible nature of the process described by (1.1).

The nonlinear wave equation

$$\left\{\frac{\partial^2}{\partial t^2} - \triangle + 1 - f(\varphi^*\varphi)\right\}\varphi(x, t) = 0 \tag{0.1}$$

under certain constraints on the function f, has a solution of the form

$$\varphi(x, t, E) = a(x, E)\, e^{iEt} \tag{0.2}$$

where the function $a(x, E)$ decreases exponentially as $|x| \to \infty$.

The operation of the Lorentz transformation on a wave packet $\phi(x, t, E)$ at rest yields the moving wave packet

$$\varphi(x, t, E, \beta) = \exp\left(i\frac{Et - \beta x}{\sqrt{1-\beta^2}}\right) a\left(\frac{x - \beta t}{\sqrt{1-\beta^2}}, E\right) \qquad (\beta \text{ — the velocity}) \tag{0.3}$$

which also satisfies equation (0.1) because of its Lorentz invariance.

A question which it is natural for physicists to discuss repeatedly is [1]: May the solutions (0.2) be interpreted as particles?* From this viewpoint, what must first be

*The solution of (1.1) may be given physical meaning by comparing it (because of (0.1)) to conservation of the four-vector of the momentum's energy

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clarified is: What is the result of the collision of two moving wave packets, $\phi_1 = \phi(x, t, E, \beta)$ and $\phi_2 = \phi(x, t, E, -\beta)$, say. Let the solution $\phi(x, t)$ of (0.1) be determined by the asympototic condition $\phi(x, t) - \phi_1, \phi(x, t) - \phi_2 \to 0$ as $t \to -\infty$; it is necessary to establish whether an analogous asymptotic condition will be satisfied as $t \to +\infty$.

There is, however, a simpler question, which is also essential if the above-mentioned physical interpretation of the solution (0.2) is to be possible (following[1], we shall designate this solution as particle-like); this is the question of the stability of the solution (0.2) with respect to small changes in the initial conditions, i.e. the question of Liapunov stability. The given work [2] is also devoted primarily to the investigation of this question.

The simplest case of a wave equation with one space coordinate is considered in detail in section 2; complications which arise in the case of three space degrees of freedom are discussed in section 4. Let us take the simplest kind of function

$$f = (\varphi^*\varphi)^n, \qquad n > 0 \tag{0.4}$$

The principal novelty herein is the generalization of the known statement on the stability (instability) of the equilibrium position in which the potential energy has a minimum (saddle point) as applied to the 'equilibrium position' (0.2), i.e. the simplest periodic solutions. The proposed extension is very simple: it is based entirely on the fact that (as is easy to verify) the 'point' (0.2) achieves the extremum to the energy functional (1.1) when a surface of constant charge (1.2) passes through it. The idea of the similarity of (0.1) to the systems of equations in mechanics plays an essential part herein. The resulting hypotheses on the behaviour of $\phi(x, t)$ as $t \to \infty$ are expounded in section 3. We take the opportunity to thank V.V. Babnikov, V.K. Mel'nikov, M.A. Markov, Iu. P. Rybakov and Ia.P. Terletskii for interest in the research.

1. Of the number of integrals of the motion defined by equation (0.1) let us write down the energy E^* and the charge Q:

$$E^* = \int_{-\infty}^{+\infty} dx \{\dot\varphi^*\dot\varphi + \nabla\varphi^*\nabla\varphi + \varphi^*\varphi - F(\varphi^*\varphi)\} \qquad \left(F(z) = \int_0^z f(x)\,dx\right) \tag{1.1}$$

$$Q = -\frac{i}{2}\int_{-\infty}^{+\infty} dx\,(-\dot\varphi^*\varphi + \dot\varphi\varphi^*) \tag{1.2}$$

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$$p_i = \int dx T_{i4}$$

here T_{ik} is the tensor of the momentum-energy density (see, [2], etc, for example). Eybakov [3] also investigated the stability of the solution (1.2), but did not succeed in obtaining a specific result.

We shall denote the problem of seeking the solution $\varphi(x, t)$ of (0.1) with the initial conditions $\varphi(x, 0) = \varphi_0(x)$, $\varphi^{\cdot}(x, 0) = \psi_0(x)$ as $P(\varphi_0, \psi_0)$. Later we shall also need equation (0.1) in integral form

$$\varphi(x, t) = \varphi_0(x, t) + G[f(\varphi^*\varphi)\varphi](x, t) \tag{1.3}$$

Let us present an explicit expression for the operator G for the one-dimensional case

$$G\eta(x, t) = \frac{1}{2}\int_0^t d\tau \int_{\tau-t}^{t-\tau} d\xi\, J_0\left(\sqrt{(t-\tau)^2 - \xi^2}\right) \eta(x-\xi, \tau) \tag{1.4}$$

The function $\varphi_0(x, t)$ in (1.3) is a solution of the problem $P_0(\varphi_0, \psi_0)$, i.e. of the linear equation

$$(\partial^2 / \partial t^2 - \triangle + 1)\varphi_0(x, t) = 0 \tag{1.5}$$

with the initial condition (φ_0, ψ_0). We shall denote the problem of seeking the solution of (1.3) by $L(\varphi_0, \psi_0)$.

Let us also note the following notation used later

$$J_n(\varphi) = \int |\varphi|^n dx, \quad R_E(\varphi) = \int \{\nabla\varphi^*\nabla\varphi + \varphi^*\varphi(1 - E^2)\}\, dx$$
$$\omega(p) = \sqrt{p^2 + 1} \tag{1.6}$$

2. In order to determine the function $a(x, E)$ we obtain the equation

$$\left\{-\frac{d^2}{dx^2} + 1 - E^2 - (a^*a)^n\right\} a(x, E) = 0 \tag{2.1}$$

It is easy to see that for $E^2 < 1$ this equation has a unique (to the accuracy of the transformation $a(x, E) \to a(x + \alpha, E)e^{i\beta}$), solution $a(x, E)$ tending to zero as $|x| \to \infty$ where

$$a(x, E) = (1 - E^2)^{\frac{1}{2n}} b\left(x\sqrt{1 - E^2}\right)$$
$$b(x) = O(e^{-|x|}) \quad \text{for } |x| \to \infty, \quad b(-x) = b(x) \tag{2.2}$$

Substituting (2.2) into (1.1) and (1.2) we find

$$E^* = E^*(E) = \frac{2}{n+2}(1 - E^2)^{\frac{1}{n} - \frac{1}{2}}(n + 2E^2) J_2(b) \tag{2.3}$$

$$Q = Q(E) = (1 - E^2)^{\frac{1}{n} - \frac{1}{2}} J_2(b) \tag{2.4}$$

In deriving (2.3) we have used the relationship

$$\int_{-\infty}^{+\infty} dx\ [2z\ (1-E^2) - F\ (z) - zf\ (z)] = 0 \qquad (z = [a\ (x,\ E)]^2) \tag{2.5}$$

It may easily be derived from the condition that the integral of the space components of the energy-density tensor vanishes, say (see footnote on pages 497 and 498) which corresponds to the solution (0.2) (Laue theorem [2]).

2.1. It is convenient to formulate the local existence theorem as follows.

Theorem 2.1. If $|\phi_0\ (x,\ t)| < M_0$ for $t \geqslant 0$, the solution $\phi\ (x,\ t)$ of the problem $L\ (\varphi_0,\ \psi_0)$ may be constructed for $0 < t < T,\ M_0 T = \alpha\ (n) > 0$ by iterations of the form

$$\varphi_{n+1}\ (x,\ t) = \varphi_0\ (x,\ t) + G\ [f\ (|\varphi_n|^2)\ \varphi_n]\ (x,\ t) \tag{2.6}$$

Here $\phi_0\ (x,\ t)$ is the solution of the problem $P_0\ (\varphi_0,\ \psi_0)$, i.e. of (1.5).

The solution $\phi\ (x,\ t)$ will be : *a*) unique ; *b*) bounded, i.e.

$$|\varphi\ (x,\ t)| < \beta\ (n)\ M_0 \quad \text{for} \quad t < T$$

c) have continuous derivatives to second order inclusive and satisfy the equation $P\ (\varphi_0,\ \psi_0)$, if the functions $\varphi_0\ (x)$, $\psi_0\ (x)$ and $d\varphi_0\ /\ dx$ have continuous first derivatives.

Proof. The boundedness of the iteration is established by comparing the estimates for $|\phi_n(x,\ t)|$ with the expansion $x = y + y^{1+2n} + \ldots$ ($|y| < \alpha\ (n)$) of the root of the equation $x = y + x^{1+2n}$. The rest of the proof is standard. See also [4], where an equation similar to (0.1) is examined.

Corollary 2.1. If there is a number M independent of τ, for the given functions $\varphi_0\ (x)$, $\psi_0\ (x)$ (continuous together with the derivatives $d\varphi_0\ /\ dx$, $d\psi_0\ /\ dx$, $d^2\varphi_0\ /\ dx^2$) such that it follows from the existence of a solution $\phi\ (x,\ t)$ of the problem $P\ (\varphi_0,\ \psi_0)$ for $0 < t < \tau$ that $|\phi\ (x,\ t)| < M$ for $0 < t\ \tau$, then the solution $\phi\ (x,\ t)$ exists on the whole half-axis $t > 0$.

The proof is obvious. Corollary 2.1 guarantees the existence of a solution to the problem $P\ (\varphi_0,\ \psi_0)$ for motion around a stable generalized equilibrium position $\phi\ (x,\ t,\ E)$ (section 2.4).

2.2. Let $q = Q\ (E)$ be the quantity of charge corresponding to the solution $\phi\ (x,\ t,\ E)$. The functional

$$V_q\ (\varphi) = \frac{q^2}{J_2\ (\varphi)} + \int [\nabla\varphi^*\nabla\varphi + \varphi^*\varphi - F\ (\varphi^*\varphi)]\ dx \tag{2.7}$$

obtained by minimizing the functional (1.1) of the energy $E^*\ (\phi,\ \dot\phi)$ with respect to the 'velocity' $\dot\phi$ under the condition $Q\ (\varphi,\ \dot\varphi) = q$

$$E^* (\varphi, \varphi^{\cdot}) \geqslant V_q (\varphi) \quad \text{for} \quad Q (\varphi, \varphi^{\cdot}) = q \tag{2.8}$$

plays a fundamental part in the investigation of the stability of the solution $\phi (x, t, E)$.

It is easy to see that the point $a (x, E)$ (see (0.2)) is an extremum for the functional $V_q(\phi)$. After this, it is at once clear from (2.8) (by analogy with mechanics) that if the 'point' $\phi = a (x, E)$ is a minimum of $V_q(\phi)$, the solution $\phi (x, t, E)$ is 'stable'. Before giving an exact formulation, let us recall how we prove the stability of an equilibrium position at which the potential energy has a minimum.

Lemma 2.1. Let (1) the function $x (t)$ satisfy the equation $x^{\cdot\cdot} + \text{grad}\, U = 0$; furthermore, (2) let positive numbers α and ρ exist such that

$$\alpha \sum_i x_i^2 \leqslant U (x) - U (0) \quad \text{for} \quad \Sigma x_i^2 < \rho$$

Then from compliance with the inequalities

$$\delta < \rho, \qquad \Sigma [x_i (t)]^2 < \delta$$

$$\Sigma [x_i^{\cdot} (t)]^2 + U [x (t)] - U (0) < \alpha\delta \quad \text{and } t = 0 \tag{2.9}$$

compliance for all $t > 0$ follows.

Proof. Let

$$\delta < \rho' < \rho, \quad \Sigma [x_i (t_0)]^2 = \rho', \quad t_0 > 0$$

Since $\rho' < \rho$ then $U (x (t_0)) - U (0) \geqslant \alpha\rho' > \alpha\delta$, which contradicts (2.9) because of the conservation of energy.

2.3. The situation is more complicated in the problem under consideration. The functional $V_q(\phi)$ has an extremum at each point of the manifold S_E defined as

$$(S_E) \qquad \varphi = e^{i\beta} a (x + \gamma, E), \qquad 0 \leqslant \beta < 2\pi, \; -\infty < \gamma < +\infty \tag{2.10}$$

The situation will be similar when the potential energy in mechanics is independent of one of the coordinates. Correspondingly, the stability of the solution $\phi (x, t, E)$ may hold only in the sense that (1) if $\phi (x, t)$ is a solution of (0.1); (2) the least distance between the point $\phi (x, t)$ and points of the manifold (2.10) is small for $t = 0$, then this distance will also be small for all $t > 0$.

Let us take the value of the functional $R_E(\phi - \phi')$ (see (1.6)) as the 'distance' $l (\phi, \phi')$ between the points $\phi (x)$ and $\phi'(x)$. Such a definition of the distance is adequate for the problem under consideration: for an unfortunate choice of the definition of the distance the stability in the above-mentioned sense will not hold (for example, for $l (\varphi, \varphi') = J_2 (\varphi - \varphi'))$.

2.4. Now an exact formulation may already be given. For $\varphi^{\circ} (x) \in S_E$ the functional $R_E(\phi - \phi^0)$ has a lower bound; it is easy to see that it reaches its lower boundary on S_E (if $\phi(x) \to 0$ as $|x| \to \infty$). Let this occur for $\phi^0(x) = D_E\phi(x)$.

$$\varphi(x) = D_E\varphi(x) + B_E\varphi(x) \tag{2.11}$$

The following will be the analog to Lemma 2.1 which will be suitable later.

Lemma 2.2. If numbers $\alpha > 0$ and $\beta < 0$ are found such that

$$\alpha R_E[B_E\varphi] < V_q(\varphi) - V_q(D_E\varphi) \qquad \text{for} \quad R_E(B_E\varphi) < \rho \tag{2.12}$$

and $\phi(x, t) = \phi_t$ is the solution of (0.1) with the same value of the charge $(x, t, E) \equiv \varphi_{E,t}$, then upon compliance with the inequalities

$$R_E(B_E\varphi_t) < \delta, \qquad E^*(\varphi_t, \dot\varphi_t) - E^*(\varphi_{Et}, \dot\varphi_{E,t}) < \alpha\delta \tag{2.13}$$

at $t = 0$ compliance with the inequality $R_E(B_E\varphi_t) < \delta$ for all $t > 0$ will follow for $\delta < \rho$.

The proof duplicates the proof of Lemma 2.2; it need only be taken into account that the functional $R_E(B_E\phi_t)$ is continuous in t (if the solution ϕ_t is continuous*).

It now remains to elucidate whether the inequality (2.12) is valid for the given solution $\phi(x, t, E)$.

2.5. Let us consider the functional $V_q[a(x, E) + \alpha v(x)]$; we have

$$V_q(a + \alpha v) = V_q(a) + \alpha^2 A(a, v) + C(a, \alpha, v)$$
$$\left(A(a, v) = \frac{d^2}{d\alpha^2} V_q(a + \alpha v)\big|_{\alpha=0}\right) \tag{2.14}$$

Lemma 2.3. For any number n (see (0.4)), numbers m $(2 < m \leqslant 3)$ and H may be given such that

$$|C(a, \alpha, v)| < \alpha^m H \rho^m \quad \text{for} \quad R_0(v) = \rho^2 \tag{2.15}$$

Proof. Terms of the functional (2.7)

$$C_1 = q^2/J_2(\varphi), \qquad C_2 = \int dx\, F(\varphi^*\varphi) \tag{2.16}$$

yield a contribution to C.

In order to estimate C_2, let us use the inequality ($k > 1$ in the case under consideration)

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* The existence of the solution ϕ_t for all $t > 0$ results from Corollary 2.1; as long as the solution exists for $0 < t < T$ (Theorem 2.1), the first inequality of (2.13) is satisfied, which (see APPENDIX (B.1) below) affords the possibility of using Corollary 2.1. Existence of continuous derivatives $d\psi_0/dx$, $d^2\varphi_0/dx^2$: should be required here; in order to avert a discussion of the convergence of the integrals in (1.1) and (1.2), the functions ϕ_0 and ψ_0 may be assumed finite.

$$|(1 - 2x + x^2 + y^2)^k - 2kx + k(x^2 + y^2) + {}^1/_2 k(k-1)\,4x^2| < \\ < h\,[(x^2 + y^2)^{m/2} + (x^2 + y^2)^k] \tag{2.17}$$

Here $m = 2k$ for $k < 3/2$ and $m = 3$ for $k \geqslant 3/2$. Let us substitute (0.4) into (2.16) with

$$x = \alpha \operatorname{Re}[v / a(x, E)], \qquad y = \alpha \operatorname{Im}[v / a(x, E)]$$

Taking into account the boundedness of the function $a(x, E)$ we obtain

$$|C(a, \alpha, v)| < h\,[\alpha^m \operatorname{Im}(v) + \alpha^{2k} J_{2k}(v)] \tag{2.18}$$

Because of (B.1), the estimate (2.15) for C_2 thus follows; it is established eve n more simply for C_1. The Lemma is proved.

Hence, in order to elucidate whether there is an inequality (2.12) for the given solution $\phi(x, t, E)$ it is necessary to know the sign of the lower bound λ of the functional

$$\Lambda(\varphi) = A(D_E\varphi, B_E\varphi) / R_E(B_E\varphi) \tag{2.19}$$

An appropriate investigation is made in APPENDIX A. Its result is that

$$\lambda \gtrless 0 \qquad \text{for} \quad E^2 \gtrless {}^1/_2 n \tag{2.20}$$

Let us note that this formula, which defines the stability region, may 'be obtained' from the following guiding considerations: The 'points' $a(x, E)$, where the number E is the root of the equation $q = Q(E)$ (see (2.4)), are stationary points of the functional $V_q(\phi)$. For $n \leqslant 2$ this equation has two roots; for $n > 2$, only one. Moreover, the functional $V_q(\phi)$ has improper 'stationary points', particularly the 'point' ϕ_1:

$$E \int |\varphi_1|^2 dx = q, \qquad \int |\nabla\varphi_1|^2 dx = \int |\varphi_1|^{2+2n} dx = 0 \tag{2.21}$$

Thus the inequality $E^2 > \frac{1}{2}n$ is the condition that of all the stationary points of the functional $V_q(\phi)$, it should take the least value at the point $\phi(x, t, E)$.

2.6. Thus, for $1 > E^2 > {}^1/_2 n$ the solution $\phi(x, t, E)$ (see (0.2)) of equation (0.1) (the function f is defined by (0.4)) is stable in the sense of Lemma (2.2) see (B.I.) also). Now, if $E^2 < \frac{1}{2}n$, then the point $a(x, E)$ is a saddle point for the functional $V_q(\phi)$; as yet it may only be said that in this case it is impossible to prove the stability of the solution $\phi(x, t, E)$ by the method described. However, sufficiently reliable, although not rigorous, resonings are presented in APPENDIX A, which show that for $E^2 < \frac{1}{2}n$ the solution $\phi(x, t, E)$ is unstable to a first approximation* (hence, the instability of $\phi(x, t, E)$ apparently follows). Hence, a very definite connection exists between the stability of the solution (0.2) and the nature of the extremum which the energy functional has at the point $\phi(x, t, E)$.

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* For $E = 0$ this result is obtained perfectly rigorously. Hobart [5] also considered the $E = 0$ case.

3. Let us examine the real solution $\phi(x, t)$; let us put $n = 1$ in (0.4) for definiteness.

Let us consider the functional of the potential energy

$$U(\varphi) = \int dx\, [|\nabla\varphi|^2 + |\varphi|^2 - {}^1\!/_4 |\varphi|^4]$$

on the 'line $\phi = c\psi$; as c increases from zero $U(c\psi)$ first increases, reaches a maximum

$$M(\psi) = [R_0(\psi)]^2 / [2J_4(\psi)]$$

and then decreases. It follows from (B.1) that the lower boundary α of the functional $M(\psi)$ is positive. Let us define the domain K by the conditions

$$U(\varphi) < \alpha, \qquad \frac{d}{dc} U(c\varphi)\big|_{c=1} > 0.$$

Using Corollary 2.1, it is easy to see that the solution of the problem $P(\varphi_0, \psi_0)$ exists for $\varphi_0 \in K$ for $t > 0$ if

$$\int |\psi_0(x)|^2 dx + U(\varphi_0) < \alpha$$

The corresponding motion is similar to finite motion in a potential well in mechanics; the difference, in particular, is that the number of our degrees of freedom is infinite, hence, the phase volume accessible for the motion is infinite, and hence, there is no theorem of return (to a small neighborhood of the initial point after a sufficiently long time) [6]. In this sense the motion described by (0.1) is irreversible.

Let us turn in (0.1) from $\phi(x, t)$ to its Fourier transform $\Phi(p, t)$. The equation thus obtained will evidently describe an infinite system of coupled oscillators. According to existing conceptions, the evolution of such a system should proceed in the direction of an approach to statistical equilibrium, i.e., towards the establishment of a uniform distribution in the kinetic energy

$$T = \int |\dot{\Phi}|^2 dp \tag{3.1}$$

between the degrees of freedom so that the existing domain of integration

$$|p| < N(t)$$

in (3.1) will increase unboundedly as t grows, and the integrand (in some sense) ceases to depend on p and approaches zero within this domain.

Hence $\Phi(p, t) \to 0$ (as $t \to \infty$) and the nonlinearity in the equation for $\Phi(p, t)$ becomes just a small addition so that

$$\Phi(p, t) = A(p, t) \cos[\omega(p)\, t + \delta]$$

where A and δ are slowly varying functions of time.

According to the above, it should be expected that as $t \to \infty$ the integral

$$\int dp\ [\omega\ (p)]^{2+\epsilon}\,|A\ (p,\ t)\,|^2$$

will increase without limit for $\epsilon > 0$, remain bounded for $\epsilon = 0$ and approach zero for $\epsilon < 0$. Hence

$$\int |A\ (p,\ t)\,|\,dp \leqslant \left\{\int [\omega\ (p)]^{-2}dp \int |\,\omega\ (q)\ A\ (q,\ t)\,|^2 dq\right\}^{1/2}$$

follows because of the inequality

$$\varphi\ (x,\ t) \to 0 \qquad \text{as} \quad t \to \infty \tag{3.2}$$

3.1. The proposed picture of the behavior of $\phi\,(x,\,t)$ as $t \to \infty$, given in section 3, is connected with such complicated representations as the postulate on the equidistribution of energy in the degrees of freedom. Meanwhile, there is an incomparably simpler mechanism leading to (3.2), namely, that which guarantees (3.2) in the case of a linear equation.

3.2. In the case of a 'quasi-finite' complex solution of (0.1), i.e. for motion similar to the stable 'equilibrium position' $\phi\,(x,\,t,\,E)$, the presence of several particle-like solutions

$$\varphi\ (x,\ t) \sim \sum_k \varphi\ (x,\ t,\ E_k,\ \beta_k) + \varphi_3$$

carrying the whole charge may be expected in the asymptotic $\phi\,(x,\,t)$ in addition to the part ϕ_3 described in section 3, where one of these solutions will turn out to be similar (in the sense of Lemma 2.2) to $\phi\,(x,\,t,\,E)$. For $n > 2$ the charge is diffused, see minimum (2.21) of the functional $V_q(\phi)$.

4. In general, the set of particle-like solutions is essentially richer in the three-dimensional than in the one-dimensional case; there are radially-symmetric solutions without nodes, with one, two, etc. nodes [1]. Furthermore, solutions of the form

$$a\ (\mathbf{x},\ E) = \alpha_m\ (r,\ \theta,\ E)\ e^{im\varphi}$$

are probably possible.

Here r, θ and ϕ are the spherical coordinates of the point $\mathbf{x}$.

Moreover, similarly to (2.5), a formula (valid for any function f)

$$\int d^3x\ \{2z\ (1-E^2) - 3F\ (z) + zf\ (z)\} = 0, \qquad z = |\,a\ (x,\ E)\,|^2 \tag{4.1}$$

may be obtained in the three-dimensional case.

It hence follows that equation (0.1) may not have particle-like solutions; such is the situation, for example*, for $f = |\varphi|^{2n}$, $n \geqslant 2$.

* In the paper 'Monozhestvo reshenii kraevoi zadachi dlia nekotorykh uravnenii

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4.1. The properties of the manifold $R_E(\phi) = \rho$ change in an essential manner with the transition to the three-dimensional case; the inequality (B.1) is replaced* by the inequality (B.2).

The proof of the existence theorem is correspondingly complicated; for $n < 1$ it can be carried out by the method in [8] (see [9] also): the iterations this time converge to the norm†

$$\|\varphi\| = \int d^3x\, [|\varphi^{\cdot}|^2 + |\nabla\varphi|^2 + |\varphi|^2]$$

4.2. An investigation of the quadratic functional (2.19) is carried out in exactly the same manner as in the one-dimensional case: this time equation (A.6) yields $E^2 = 3n/2$.

The solution $\phi(x, t, E)$ is stable for $1 > E^2 > 3n/2$, if (A.4) has only one eigenvalue less than unity, $m_0 = 1$, for $\alpha = 0$; if (A.4) has more than one, $m_0 > 1$, such eigenvalues for $\alpha = 0$, then it has at least one eigenvalue $\lambda'_0(\alpha)$ less than one for all values $\alpha > 0$.

Hence, for $m_0 > 1$ the solution (x, t, E) is apparently unstable. Evidently $m_0 > 1$ for solutions $\phi(x, t, E)$ with nodes (see [10], chapter VI, section 1).

A numerical computation, made for a radially-symmetric solution without nodes for $n = 1$ (see (0.4)), showed $m_0 = 1$: however, a proof that $m_0 = 1$ for all (or at least for radially-symmetric) nodeless solutions has been unsuccessful.

APPENDIX A. Let us consider the problem of seeking the minimum of the functional

$$\Lambda\,[u(x)] = A\,[a(x, E),\ u(x)]/R_E(u) \tag{A.1}$$

in the class of functions $u(x)$ such that

$$\int dx\, a(x, E)\left[1 - E^2 - \frac{d^2}{dx^2}\right] \mathrm{Im}\, u(x) = \int dx\, \frac{\partial a(x, E)}{\partial x}\left[1 - E^2 - \frac{d^2}{dx^2}\right] \mathrm{Re}\, u(x) = 0$$

(see the definition of the function $B\phi(x)$ at the beginning of section 2.4).

Minimization of the functional $\Lambda(u)$ leads to the equation

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fiziki' (Set of solutions of boundary value problems for some equations of mathematical physics, Preprint OIIaI P-1682, 1964) V.P. Shirikov proved the existence of radially-symmetric solutions with any number of nodes for $n < 3/2$. Nehari [7] proved the absence of radially-symmetric particle-like solutions for $n > 2$; the absence of any particle-like solutions follows from (4.1). This result is a consequence of the fact that the number α of section 3 is zero for $n > 2$.

* Hence, a Lemma of the form (2.5) may only be proved for $n < 2$.

† Proof of the existence for $2 > n > 1$ is an interesting, though unsolved problem.

$$\left\{-\frac{d^2}{dx^2}+1-E^2-(2n+1)\,a^{2n}(x,E)\,\lambda_1'\right\}u_1(x)+$$
$$+\lambda_1'\,[Q(E)]^2\,\{J_2.[a(x,E)\}^{-3}\,a(x,E)\int ds u_1(s)\,a(s,E)=0 \quad \text{(A.2)}$$

$$\left\{-\frac{d^2}{dx^2}+1-E^2-\lambda_2' a^{2n}(x,E)\right\}u_2(x)=0 \quad \text{(A.3)}$$

$$u_1=\mathrm{Re}\,u(x),\quad u_2=\mathrm{Im}\,u(x),\quad \lambda'=(1-\lambda)^{-1}\qquad (\lambda=\Lambda\,[u_1+iu_2])$$

Note that λ' is a monotonically increasing function of λ, where $\lambda'=1$ for $\lambda=0$. Let us first turn our attention to the fact that (A.2) and (A.3) have the eigenfunctions

$$u_1=u_{10}=\frac{\partial a(x,E)}{\partial x},\qquad u_2=u_{20}=a(x,E)$$

with the eigenvalues $\lambda'=\lambda_2'=1$. Hence, it is easy to see that the eigenfunctions of equations (A.2) and (A.3) with eigenvalues $\lambda'\neq 1$ satisfy conditions (A.1). On the other hand, the functions u_{10} and u_{20} do not satisfy these conditions. Assuming the eigenfunctions of equations (A.2) and (A.3) to form a complete system, we see that the lower bound of the functional (2.19) is determined by the lowest eigenvalue of (A.2) and (A.3) (the eigenfunctions u_{10} and u_{20} and their eigenvalues are not taken into account in the computation). The function $a(x,E)$ has no zeros, hence, the eigenvalue $\lambda'=1$ is lowest for (A.3). Furthermore, the substitution $x\sqrt{1-E^2}=y$ (see (2.2)) reduces (A.2) to the form

$$\left\{-\frac{d^2}{dy^2}+1-\lambda_1'(2n+1)\,b^{2n}(y)\right\}w(y)+\alpha\lambda_1' b(y)\,[J_2(b)]^{-1}\int w(z)\,b(z)\,dz=0$$
$$\left(\alpha=\frac{4E^2}{1-E^2}\right) \quad \text{(A.4)}$$

Differentiating with respect to α we find

$$\frac{d\lambda_1'}{d\alpha}=\frac{\lambda_1'}{J_2(b)\,R_0(w)}\left[\int ds\,b(s)\,w(s)\right]^2\geqslant 0 \quad \text{(A.5)}$$

For $\alpha=0$ equation (A.4) has an eigenvalue λ'_{10} less than one, and only one, because the eigenfunction $w=\partial b/\partial y$ with one zero corresponds to the eigenvalue $\lambda_1'=1$ (for $\alpha=0$ the integral in (A.4) drops out). Because of (A.5) it remains merely to explain whether the eigenvalue $\lambda'_{10}(\alpha)$ of (A.4) which takes the value λ'_{10} for $\alpha=0$, becomes one.

Differentiating (2.1) with respect to E we find

$$\left\{-\frac{d^2}{dx^2}+1-E^2-(2n+1)\,a^{2n}(x,E)\right\}\frac{[\partial a(x,E)}{\partial E}=2Ea(x,E)$$

Comparing this relation with (A.4) for $\lambda'=1$, we obtain the equation

$$E\frac{d}{dE}J_2[a(x,E)]=-J_2[a(x,E)] \quad \text{(A.6)}$$

to seek the zeros of the function $\lambda'_{10}(\alpha) - 1$. Hence, (2.20) follows.

Furthermore, let us substitute the equation $\varphi(x, t) = \varphi(x, t, E) + v$. into (0.1). Discarding higher order terms in v, we obtain a variational equation which takes the form

$$\tau = t\sqrt{1 - E^2}, \quad y = x\sqrt{1 - E^2} \qquad \text{for} \quad v = e^{\nu\tau} w(y)$$

in the variables

$$\left\{\nu^2 - \frac{d^2}{dy^2} + 1 - (2n + 1)\, b^{2n}(y)\right\} w_1(y) = \alpha\nu w_2(y) \qquad (w_1 = \operatorname{Re} w)$$
$$\left\{\nu^2 - \frac{d^2}{dy^2} + 1 - b^{2n}(y)\right\} w_2(y) = -\alpha\nu w_1(y) \qquad (w_2 = \operatorname{Im} w) \tag{A.7}$$

For $\alpha = 0$ this system has the eigenvalue $\nu_0 > 0$ (so that the solution $\phi(x, t, 0)$ is unstable in the first approximation). It is hence natural to expect that there is a region $0 \leqslant \alpha < \alpha_0$, in which the system (A.7) will have a real eigenvalue $\nu(\alpha)$, $\nu(\alpha) > 0$ for $0 \leqslant \alpha < \alpha_0$, $\nu(\alpha_0) = 0$. It is easy to see that the value α_0 is defined by the relationship (A.6).

Proof of the existence of a solution of the system (A.7) with the properties assumed above for $\alpha \neq 0$ was unsuccessful. However, the doubt arising in this connection is smoothed over by the favorable result of the numerical computation carried out for $n = 1$ (the solution of the system (A.7) was determined as a series in powers of $a(x, E)$). Let us note that equation (A.6) to seek the roots of the equation $\lambda_{10}'(\alpha) = 1$ (and the roots of the equation $\nu(\alpha) = 0$) remains valid for any function $f(\varphi^*\varphi)$).

APPENDIX B. The inequality

$$|\varphi(x)| < H\,[R_0(\varphi)]^{1/2} \tag{B.1}$$

holds for functions of one variable.

Analogously, for functions of three independent variables [11] we have

$$J_l(\varphi) < H\,[R_0(\varphi)]^{1/2l} \qquad \text{for} \quad 2 \leqslant l < 6 \tag{B.2}$$

the number H is independent of ϕ.

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